A Quick Guide for the QZ Package

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Warning: This document is written to explain the main functions of **QZ** (Chen 2013), version 0.1-0. Every effort will be made to ensure future versions are consistent with these instructions, but features in later versions may not be explained in this document.

1. Introduction

This article is to explain the **QZ** (Chen 2013), and is organized as the following. Section 2 introduces briefly background of generalized eigenvalues problem and QZ decomposition. Section 3 lists the main functions and detail Fortran functions of LAPACK library (Anderson *et al.* 1999).

2. Methods

Some details can be found on wikipedia website at

`http://en.wikipedia.org/wiki/Eigendecomposition_of_a_matrix`

for generalized eigenvalues, and at

`http://en.wikipedia.org/wiki/Schur_decomposition`

about QZ decomposition or generalized Schur form. The LAPACK (Anderson *et al.* 1999) also provides functions to solve these problems.

2.1. Generalized Eigenvalues for Pair Matrices

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $N \times N$ non-symmetric matrices which can be both in real or in complex. The goal is to find right generalized eigen vectors $\boldsymbol{v}$ such that $\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{B}\boldsymbol{v}$, or left generalized eigen vectors $\boldsymbol{u}$ such that $\boldsymbol{u}^H \boldsymbol{A} = \lambda \boldsymbol{u}^H \boldsymbol{B}$ where $\boldsymbol{u}^H$ is the conjugate-transpose of $\boldsymbol{u}$. Also, λ is called generalized eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{B}$ which obeys $\det(\boldsymbol{A} - \lambda \boldsymbol{B}) = 0$. Note that λ, $\boldsymbol{u}$, and $\boldsymbol{v}$ may be complex even $\boldsymbol{A}$ and $\boldsymbol{B}$ are in real.

Suppose $\boldsymbol{B}$ is an identity matrix $\boldsymbol{I}$, then the problem reduces to traditional eigenvalue problem. i.e. This is a special case.

2.2. QZ Decomposition for Pair Matrices

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are two $N \times N$ non-symmetric matrices which can be both in real or in complex. The QZ decomposition factorizes both matrices as

- $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Z}^\top$ and $\boldsymbol{B} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Z}^\top$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are real, or
- $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Z}^H$ and $\boldsymbol{B} = \boldsymbol{Q}\boldsymbol{T}\boldsymbol{Z}^H$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ are complex

where $\boldsymbol{Q}$ and $\boldsymbol{Z}$ are unitary and $\boldsymbol{S}$ and $\boldsymbol{T}$ are upper triangular. The unitary means $\boldsymbol{X}\boldsymbol{X}^H = \boldsymbol{I}$ if $\boldsymbol{X}$ is complex or $\boldsymbol{X}\boldsymbol{X}^\top = \boldsymbol{I}$ if $\boldsymbol{X}$ is real where $\boldsymbol{I}$ is the identity matrix.

The QZ decomposition is also called generalized Schur decomposition where $\boldsymbol{S}$ and $\boldsymbol{T}$ are the Schur form of $\boldsymbol{A}$ and $\boldsymbol{B}$. The generalized eigenvalues λ that solve the generalized eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{B}\boldsymbol{x}$ where $\boldsymbol{x}$ is an unknown nonzero vector and $\lambda_i = \boldsymbol{S}_{ii}/\boldsymbol{T}_{ii}$.

Suppose $\boldsymbol{B}$ is an identity matrix $\boldsymbol{I}$, then the problem reduces to fine $\boldsymbol{Q}$ such that $\boldsymbol{A} = \boldsymbol{QSQ}^{-1}$ for real $\boldsymbol{A}$ or $\boldsymbol{A} = \boldsymbol{QSQ}^{H}$ for complex $\boldsymbol{A}$. i.e. This is a special case.

3. Implementation

Two main functions are `geigen()` for generalized eigenvalues, and `qz()` for QZ decomposition with reordering capability. Both functions are able to deal a single matrix $\boldsymbol{A}$ or a paired matrices $(\boldsymbol{A}, \boldsymbol{B})$ in both complex and real systems. Both functions are wrapper functions for several lower level R functions `qz.*()` which are also wrapper functions via `.Call()` for C and Fortran functions to LAPACK library version 3.4.2.

LAPACK library is incorporated in **QZ** including complex*16 and double precision for complex and real systems respectively. **QZ** has functions of LAPACK and BLAS (Blackford *et al.* 2002) independently to the R's LAPACK and BLAS libraries since some functions are not available. Table 1 provides a detail lists for the `qz.*()` functions.

Table 1: **QZ** functions

Function	Wrapper	Main Input	System	Purpose
`geigen()`	`qz.zgeev`	$\boldsymbol{A}$	Complex	Generalized eigenvalues
	`qz.dgeev`	$\boldsymbol{A}$	Real	
`qz()`	`qz.zgees`	$\boldsymbol{A}$	Complex	QZ decomposition
	`qz.dgees`	$\boldsymbol{A}$	Real	
	`qz.ztrsen`	$\boldsymbol{T}, \boldsymbol{Q}$	Complex	Reordering
	`qz.dtrsen`	$\boldsymbol{T}, \boldsymbol{Q}$	Real	
`geigen()`	`qz.zggev`	$(\boldsymbol{A}, \boldsymbol{B})$	Complex	Generalized eigenvalues
	`qz.dggev`	$(\boldsymbol{A}, \boldsymbol{B})$	Real	
`qz()`	`qz.zgges`	$(\boldsymbol{A}, \boldsymbol{B})$	Complex	QZ decomposition
	`qz.dgges`	$(\boldsymbol{A}, \boldsymbol{B})$	Real	
	`qz.ztgsen`	$(\boldsymbol{S}, \boldsymbol{T}), \boldsymbol{Q}, \boldsymbol{Z}$	Complex	Reordering
	`qz.dtgsen`	$(\boldsymbol{S}, \boldsymbol{T}), \boldsymbol{Q}, \boldsymbol{Z}$	Real	

References

Anderson E, Bai Z, Bischof C, Blackford S, Demmel J, Dongarra J, Du Croz J, Greenbaum A, Hammarling S, McKenney A, Sorensen D (1999). *LAPACK Users' Guide*. Third edition. Society for Industrial and Applied Mathematics, Philadelphia, PA. ISBN 0-89871-447-8 (paperback).

Blackford L, Demmel J, Dongarra J, Duff I, Hammarling S, Henry G, Heroux M, Kaufman L, Lumsdaine A, Petitet A, Pozo R, Remington K, Whaley R (2002). "An Updated Set of Basic Linear Algebra Subprograms (BLAS)." *ACM Trans. Math. Soft.*, **28**, 135–151. URL `http://www.netlib.org/blas/`.

Chen WC (2013). "QZ: Generalized Eigenvalues and QZ Decomposition." R Package, URL `http://cran.r-project.org/package=QZ`.