Solving Differential Equations in R (book) - PDE examples

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Abstract

This vignette contains the R-examples of chapter 10 from the book:

Soetaert, K., Cash, J.R. and Mazzia, F. (2012). Solving Differential Equations in R. that will be published by Springer.

Chapter 10. Solving Partial Differential Equations in R.

Here the code is given without documentation. Of course, much more information about each problem can be found in the book.

Keywords: partial differential equations, initial value problems, examples, R.

1. The heat Equation

```
N        <- 100
xgrid    <- setup.grid.1D(x.up = 0, x.down = 1, N = N)
x        <- xgrid$x.mid
D.coeff  <- 0.01
Diffusion <- function (t, Y, parms){
   tran <- tran.1D(C = Y, C.up =  0, C.down = 1,
                D = D.coeff, dx = xgrid)
   list(dY = tran$dC, flux.up = tran$flux.up,
        flux.down = tran$flux.down)
}
Yini  <- sin(pi*x)
times <- seq(from = 0, to = 5, by = 0.01)
print(system.time(
  out <- ode.1D(y = Yini, times = times, func = Diffusion,
             parms = NULL, dimens = N)               ))
```

```
  user  system elapsed
  0.28    0.00    0.29
```

```
par (mfrow=c(1, 2))
plot(out[1, 2:(N+1)], x, type = "l", lwd = 2,
```

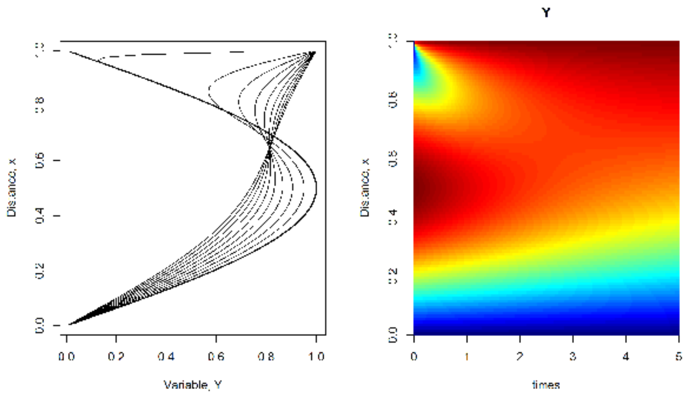

Figure 1: The solution of the heat equation. See book for more information.

```
    xlab = "Variable, Y", ylab = "Distance, x")
for (i in seq(2, length(times), by = 50))
     lines(out[i, 2:(N+1)], x)
image(out, grid = x, mfrow = NULL, ylab = "Distance, x",
     main = "Y")
```

2. The Wave Equation

```
dx    <- 0.2
xgrid <- setup.grid.1D(x.up = -100, x.down = 100, dx.1 = dx)
x     <- xgrid$x.mid
N     <- xgrid$N
lam   <- 0.05
uini  <- exp(-lam*x^2)
vini  <- rep(0, N)
yini  <- c(uini, vini)
times <- seq (from = 0, to = 50, by = 1)
wave <- function (t, y, parms) {
  u <- y[1:N]
  v <- y[(N+1):(2*N)]

  du <- v
  dv <- tran.1D(C = u, C.up = 0, C.down = 0, D = 1,
               dx = xgrid)$dC

  return(list(c(du, dv)))
}
out <- ode.1D(func = wave, y = yini, times = times,
             parms = NULL, method = "adams",
             dimens = N, names = c("u", "v"))
u <- subset(out, which = "u")
analytic <- function (t, x)
  0.5 * (exp(-lam * (x+1*t)^2 ) +exp(-lam * (x-1*t)^2) )
OutAna <- outer(times, x, FUN = analytic)
max(abs(u - OutAna))
```

```
[1] 0.002188562
```

```
outtime <- seq(from = 0, to = 50, by = 10)
matplot.1D(out, which = "u", subset = time %in% outtime,
    grid = x, xlab = "x", ylab = "u", type = "l",
    lwd = 2, xlim = c(-50, 50),
    col = c("black", rep("darkgrey", 5)))
legend("topright", lty = 1:6, lwd = 2,
      col = c("black", rep("darkgrey", 5)),
      title = "t = ", legend = outtime)
```

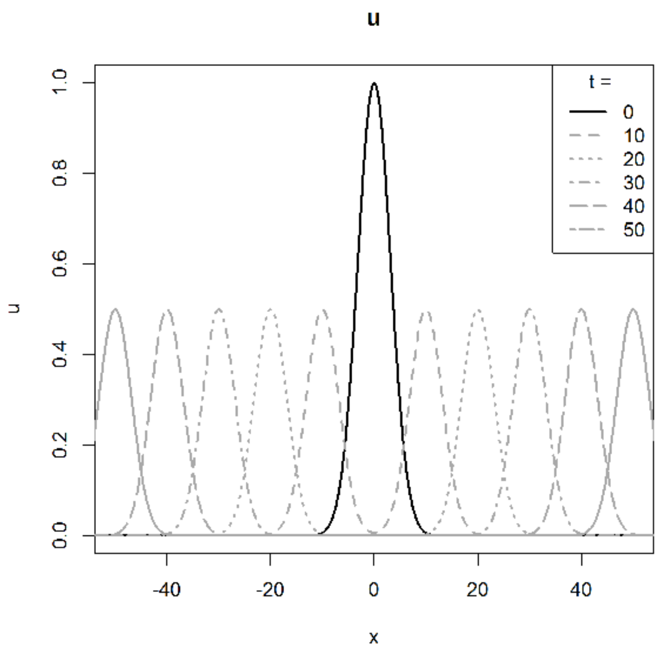

Figure 2: The 1-D wave equation. See book for explanation.

3. Laplace Equation

```
Nx <- 100
Ny <- 100
xgrid <- setup.grid.1D (x.up = 0, x.down = 1, N = Nx)
ygrid <- setup.grid.1D (x.up = 0, x.down = 1, N = Ny)
x     <- xgrid$x.mid
y     <- ygrid$x.mid
laplace <- function(t, U, parms) {
  w <- matrix(nrow = Nx, ncol = Ny, data = U)
  dw <- tran.2D(C = w, C.x.up = 0, C.x.down = 0,
             flux.y.up = 0,
             flux.y.down = -1 * sin(pi*x)*pi*sinh(pi),
             D.x = 1, D.y = 1,
             dx = xgrid, dy = ygrid)$dC
  list(dw)
}
print(system.time(
  out <- steady.2D(y = runif(Nx*Ny), func = laplace,
                parms = NULL, nspec = 1,
                dimens = c(Nx, Ny), lrw = 1e7)
))
```

```
  user  system elapsed
  0.35    0.05    0.40
```

```
w <- matrix(nrow = Nx, ncol = Ny, data = out$y)
analytic <- function (x, y) sin(pi*x) * cosh(pi*y)
OutAna <- outer(x, y, FUN = analytic)
max(abs(w - OutAna))
```

```
[1] 0.0006024049
```

```
image(out, grid = list(x, y), main = "elliptic Laplace",
     add.contour = TRUE)
```

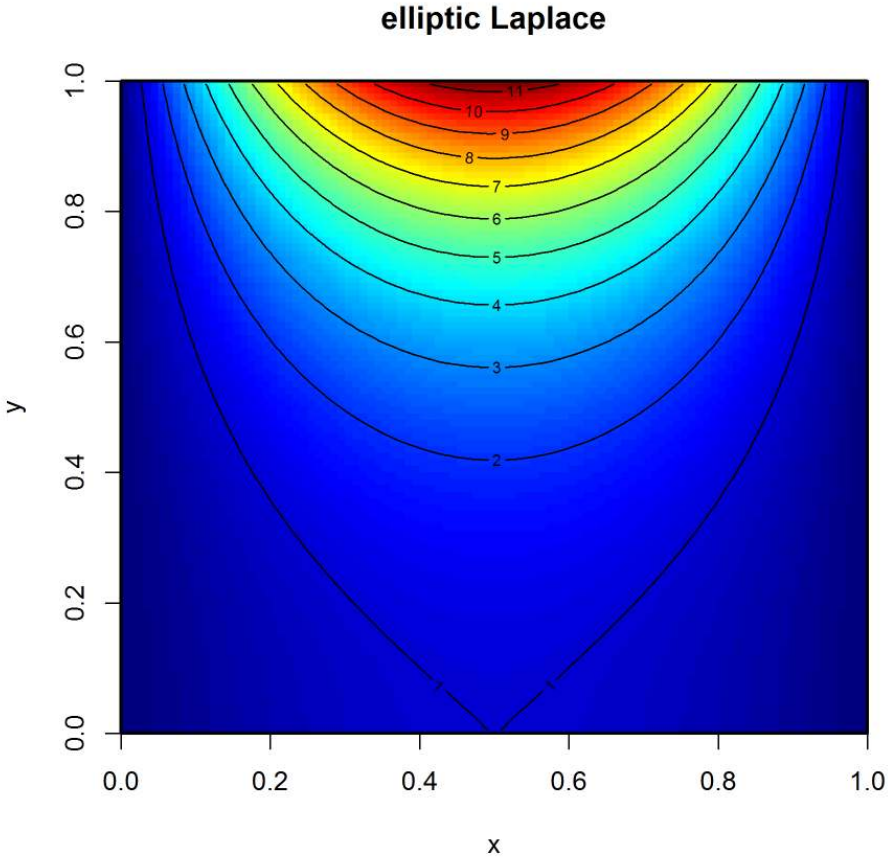

Figure 3: The laplace equation. See book for explanation.

4. The Advection Equation

```
adv.func <- function(t, y, p, adv.method)
  list(advection.1D(C = y, C.up = y[N], C.down = y[1],
                    v = 0.1, adv.method = adv.method,
                    dx = xgrid)$dC)
xgrid <- setup.grid.1D(0.3, 1.3, N = 50)
x     <- xgrid$x.mid
N     <- length(x)
yini  <- sin(pi * x)^50
times <- seq(0, 20, 0.01)
out1 <- ode.1D(y = yini, func = adv.func, times = times,
              parms = NULL, method = "euler", dimens = N,
              adv.method = "muscl")
out2 <- ode.1D(y = yini, func = adv.func, times = times,
              parms = NULL, method = "euler", dimens = N,
              adv.method = "super")
```

5. The Busselator in One Dimension

```
N    <- 50
Grid <- setup.grid.1D(x.up = 0, x.down = 1, N = N)
x1ini <- 1 + sin(2 * pi * Grid$x.mid)
x2ini <- rep(x = 3, times = N)
yini <- c(x1ini, x2ini)
brusselator1D <- function(t, y, parms) {

   X1 <- y[1:N]
   X2 <- y[(N+1):(2*N)]

   dX1 <- 1 + X1^2*X2 - 4*X1 +
         tran.1D (C = X1, C.up = 1, C.down = 1,
                 D = 0.02, dx = Grid)$dC
   dX2 <- 3*X1 - X1^2*X2 +
         tran.1D (C = X2, C.up = 3, C.down = 3,
                 D = 0.02, dx = Grid)$dC

   list(c(dX1, dX2))
}
times <- seq(from = 0, to = 10, by = 0.1)
print(system.time(
  out <- ode.1D(y = yini, func = brusselator1D,
               times = times, parms = NULL, nspec = 2,
               names = c("X1", "X2"), dimens = N)
))
```

```
  user  system elapsed
  0.23    0.00    0.25
```

```
par(mfrow = c(2, 2))
image(out, mfrow = NULL, grid = Grid$x.mid,
     which = "X1",  method = "contour")
image(out, mfrow = NULL, grid = Grid$x.mid,
     which = "X1")
par(mar = c(1, 1, 1, 1))
image(out, mfrow = NULL, grid = Grid$x.mid,
     which = "X1",  method = "persp", col = NA)
image(out, mfrow = NULL, grid = Grid$x.mid,
     which = "X1",  method = "persp", border = NA,
     shade = 0.3 )
```

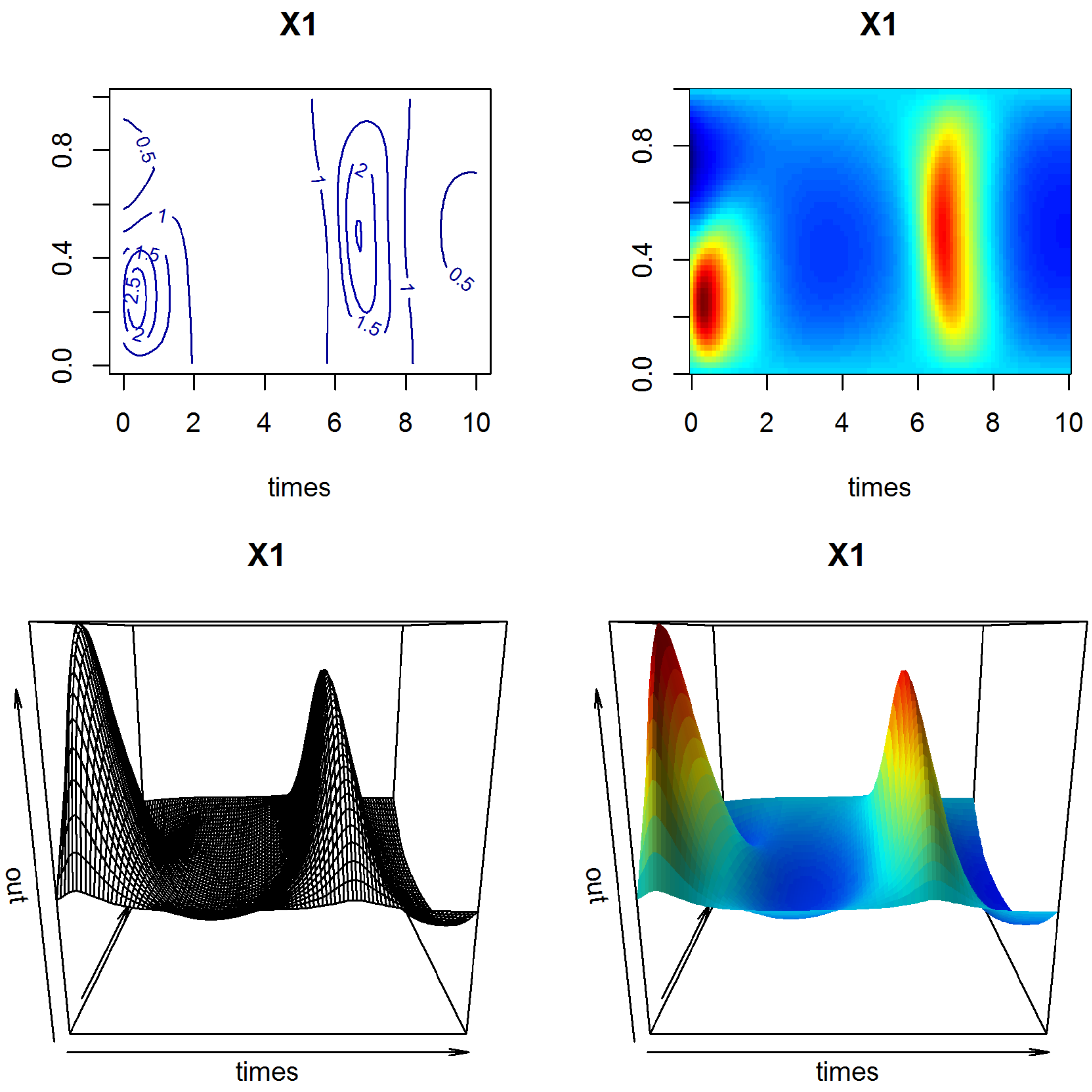

Figure 4: The 1-D Brusselator model. See book for explanation.

6. The Brusselator in 2-D

```
brusselator2D <- function(t, y, parms) {

   X1 <- matrix(nrow = Nx, ncol = Ny,
                data = y[1:(Nx*Ny)])
   X2 <- matrix(nrow = Nx, ncol = Ny,
                data = y[(Nx*Ny+1) : (2*Nx*Ny)])

   dX1 <- 1 + X1^2*X2 - 4*X1 +
         tran.2D (C = X1, D.x = D_X1, D.y = D_X1,
                  dx = Gridx, dy = Gridy)$dC
   dX2 <- 3*X1 - X1^2*X2 +
         tran.2D (C = X2, D.x = D_X2, D.y = D_X2,
                  dx = Gridx, dy = Gridy)$dC

   list(c(dX1, dX2))
}
Nx   <- 50
Ny   <- 50
Gridx <- setup.grid.1D(x.up = 0, x.down = 1, N = Nx)
Gridy <- setup.grid.1D(x.up = 0, x.down = 1, N = Ny)
D_X1 <- 2
D_X2 <- 8*D_X1
X1ini <- matrix(nrow = Nx, ncol = Ny, data = runif(Nx*Ny))
X2ini <- matrix(nrow = Nx, ncol = Ny, data = runif(Nx*Ny))
yini <- c(X1ini, X2ini)
times <- 0:8
print(system.time(
 out <- ode.2D(y = yini, parms = NULL, func = brusselator2D,
              nspec = 2, dimens = c(Nx, Ny), times = times,
              lrw = 2000000, names=c("X1", "X2"))
))
```

```
  user  system elapsed
  2.81    0.02    2.84
```

```
par(oma = c(0,0,1,0))
image(out, which = "X1", xlab = "x", ylab = "y",
     mfrow = c(3, 3), ask = FALSE,
     main = paste("t = ", times),
     grid = list(x = Gridx$x.mid, y = Gridy$x.mid))
mtext(side = 3, outer = TRUE, cex = 1.25, line = -1,
     "2-D Brusselator, species X1")
```

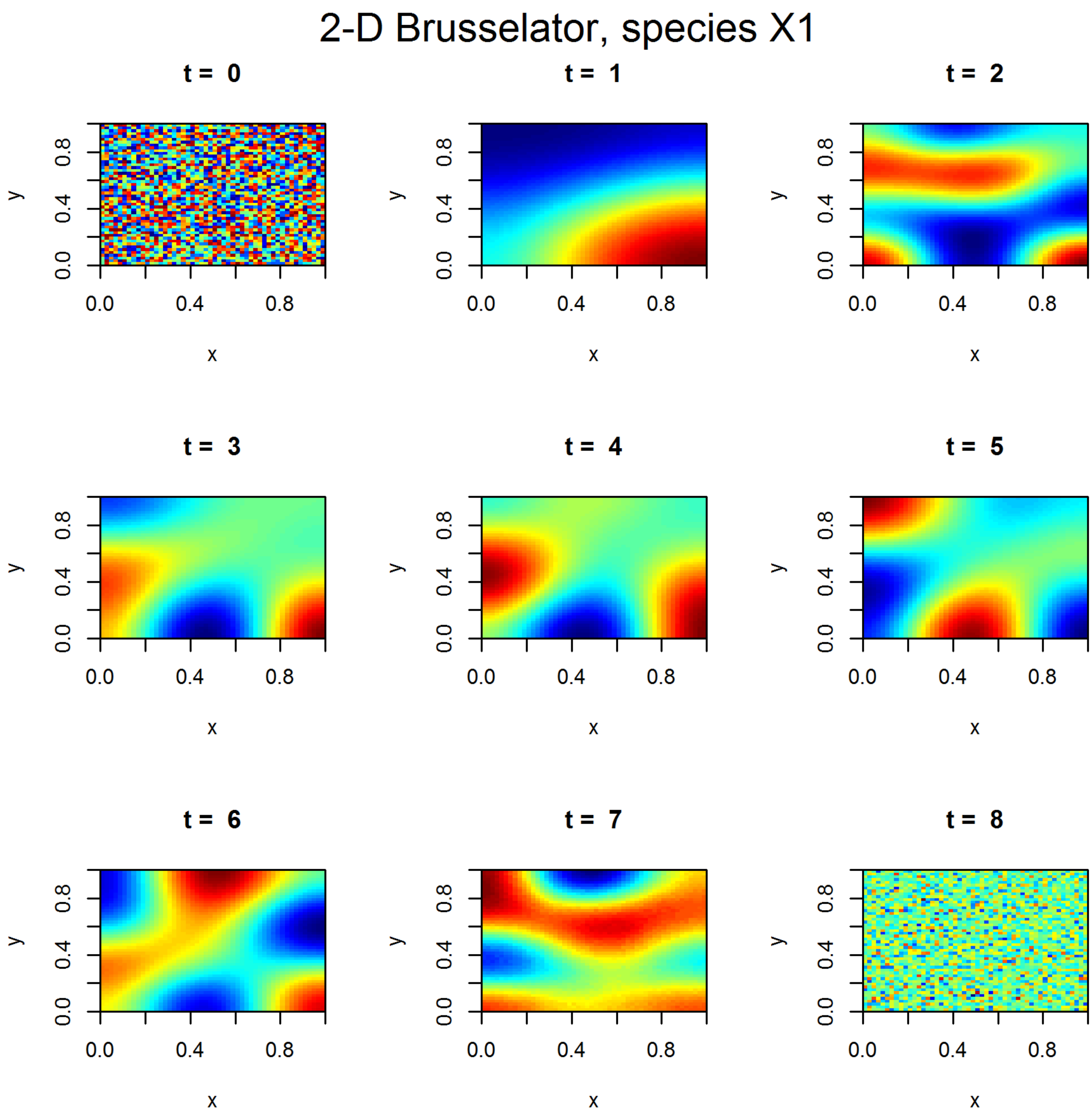

Figure 5: Solution of the 2-D Brusselator. See book for explanation.

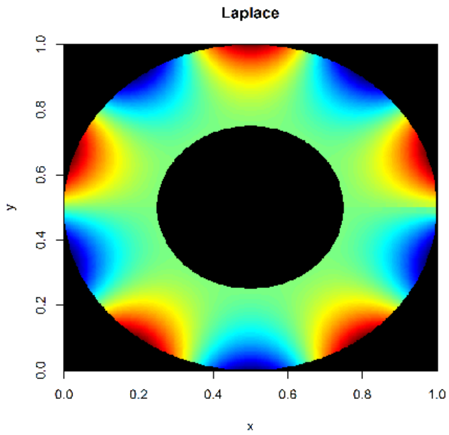

Figure 6: The Laplace equation in polar coordinates. See book for explanation.

7. The Laplace Equation in Polar Coordinates

```
Nr <- 100
Np <- 100
r          <- seq(2, 4, len = Nr+1)
theta      <- seq(0, 2*pi, len = Np+1)
theta.mid <- 0.5*(theta[-1] + theta[-Np])
Model <- function(t, C, p) {
 y = matrix(nrow = Nr, ncol = Np, data = C)
 tran <- tran.polar (y, D.r = 1, r = r, theta = theta,
             C.r.up = 0, C.r.down = 4 * sin(5*theta.mid),
             cyclicBnd = 2)
  list(tran$dC)
}
STD <- steady.2D(y = runif(Nr*Np), parms = NULL,
                func = Model, dimens = c(Nr, Np),
                lrw = 1e6, cyclicBnd = 2)
OUT <- polar2cart (STD, r = r, theta = theta,
                   x = seq(-4, 4, len = 400),
                   y = seq(-4, 4, len = 400))

image(OUT, main = "Laplace")
```

8. The Time-dependent 2-D Sine-Gordon Equation

```
Nx  <- 80
Ny  <- 80
xgrid <- setup.grid.1D(-7, 7, N=Nx)
ygrid <- setup.grid.1D(-7, 7, N=Ny)
x <- xgrid$x.mid
y <- ygrid$x.mid
sinegordon2D <- function(t, C, parms) {

   u <- matrix(nrow = Nx, ncol = Ny,
              data = C[1 : (Nx*Ny)])
   v <- matrix(nrow = Nx, ncol = Ny,
              data = C[(Nx*Ny+1) : (2*Nx*Ny)])

   dv <- tran.2D (C = u, C.x.up = 0, C.x.down = 0,
                 C.y.up = 0, C.y.down = 0,
                 D.x = 1, D.y = 1,
                 dx = xgrid, dy = ygrid)$dC - sin(u)
   list(c(v, dv))
}
peak <- function (x, y, x0 = 0, y0 = 0)
                 exp(-((x-x0)^2 + (y-y0)^2))
uini <- outer(x, y,
 FUN = function(x, y) peak(x, y, 2,2) + peak(x, y,-2,-2)
                    + peak(x, y,-2,2) + peak(x, y, 2,-2))
vini <- rep(0, Nx*Ny)
times <- 0:3
print(system.time(
out  <- ode.2D (y = c(uini, vini), times = times,
               parms = NULL, func = sinegordon2D,
               names = c("u", "v"),
               dimens = c(Nx, Ny), method = "ode45")
))

  user  system elapsed
  0.43    0.00    0.43

mr <- par(mar = c(0, 0, 1, 0))
image(out, main = paste("time =", times), which = "u",
     grid = list(x = x, y = y), method = "persp",
     border = NA, col = "grey",  box = FALSE,
     shade = 0.5, theta = 30, phi = 60, mfrow = c(2, 2),
     ask = FALSE)
par(mar = mr)
```

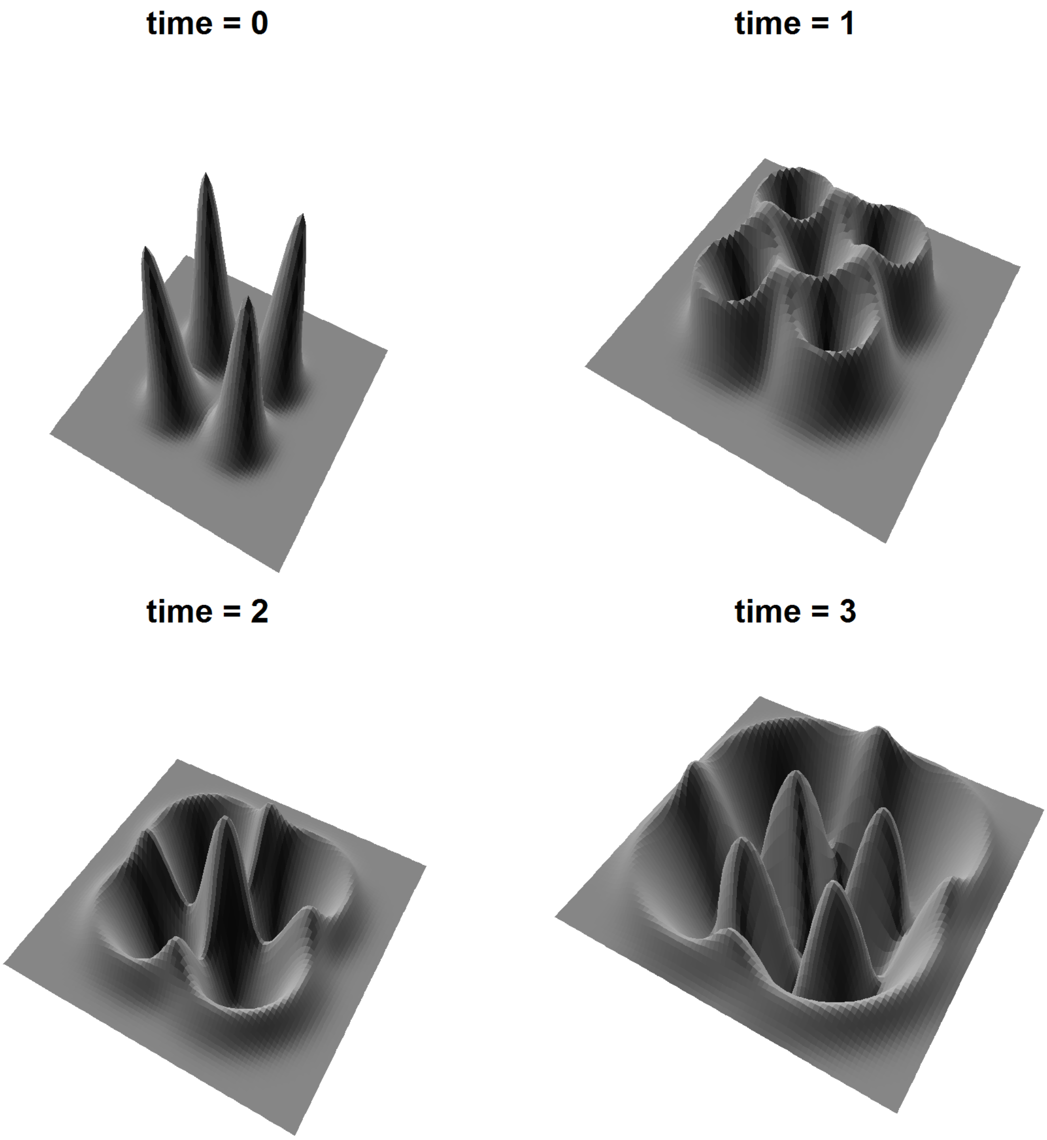

Figure 7: The 2-D sine-gordon equation. See book for explanation.

9. The Nonlinear Schrodinger Equation

```
alf <- 0.5
gam <- 1
Schrodinger <- function(t, u, parms) {
   du <- 1i * tran.1D (C = u, D = 1, dx = xgrid)$dC  +
                 1i * gam * abs(u)^2 * u
   list(du)
}
N      <- 300
xgrid <- setup.grid.1D(-20, 80, N = N)
x      <- xgrid$x.mid
c1   <- 1
c2   <- 0.1
sech     <- function(x) 2/(exp(x) + exp(-x))
soliton <- function (x, c1)
  sqrt(2*alf/gam) * exp(0.5*1i*c1*x) * sech(sqrt(alf)*x)
yini <- soliton(x, c1) + soliton(x-25, c2)
times <- seq(0, 40, by = 0.1)
print(system.time(
 out <- ode.1D(y = yini, parms = NULL, func = Schrodinger,
              times = times, dimens = 300, method = "adams")
))
```

```
  user  system elapsed
  2.18    0.03    2.28
```

```
image(abs(out), grid = x, ylab = "x", main = "two solitons")
```

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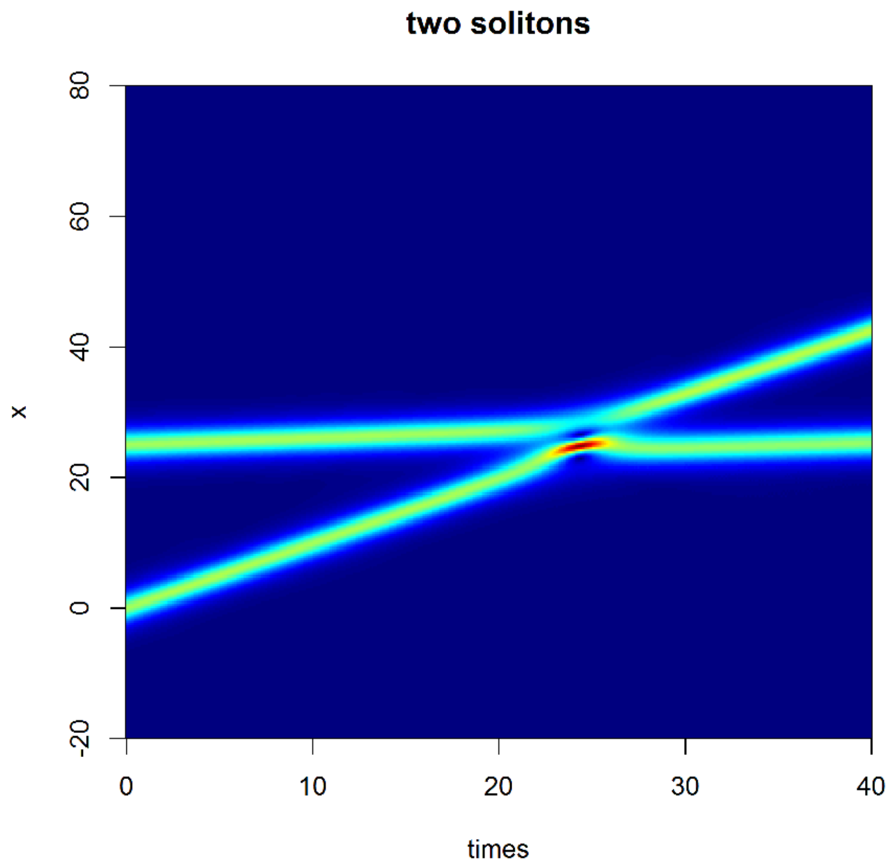

Figure 8: Solution of the Schrödinger equation. See book for explanation.