CHAPTER 6

COMPREHENSIVE EXAMPLE

6.1 Outline of Circular-Spatial Processes

This section combines the results of Chapters 2 - 5 to show that the theory and methods produce interpretable, and practical results. The processes involve:

1) Modeling the underlying trend component of the simulated circular-spatial data
2) Simulation of a CRF
3) Estimation of the spatially correlated random components of direction as the residual rotations from the trend estimate
4) Extracting the circular-spatial correlation as the cosineogram
5) Modeling the cosineogram for a smooth, continuous, and positive definite function
6) Kriging the residual rotations using the cosine model
7) Estimating the circular-spatial data
8) Plotting the circular-spatial estimate.

The R code used is located in Appendix L, Section L.9. References to the R package CircSpatial will be given. Arrow style (color, font, thickness) will be used consistently in closely related figures for the same type of information.

6.2 Simulation of a CRF

Figure 6-1, shows the trend model. Figure 6-2 shows a circular-spatial sample simulated by adding the trend model to a simulation of the von Mises CRF, $\mu = 0, \rho = \sqrt{0.5}$ transformed from a GRF with spherical covariance and range 4 (Section 5.3) using the R package CircSpatial function SimulateCRF (Appendix J, Section J.2). The green highlighted area will be enlarged in subsequent figures.

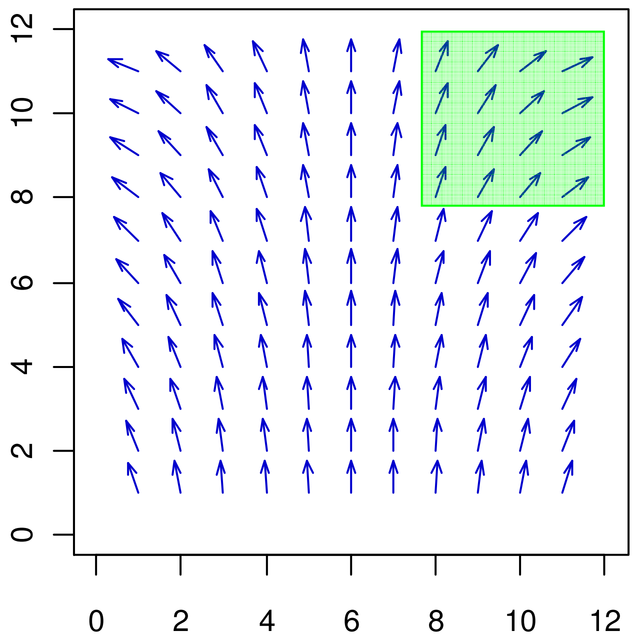

Figure 6-1. Comprehensive Example – The Trend Model, or the Underlying First Order Component of Variation. Closely related information of the green highlighted area will be enlarged in subsequent figures.

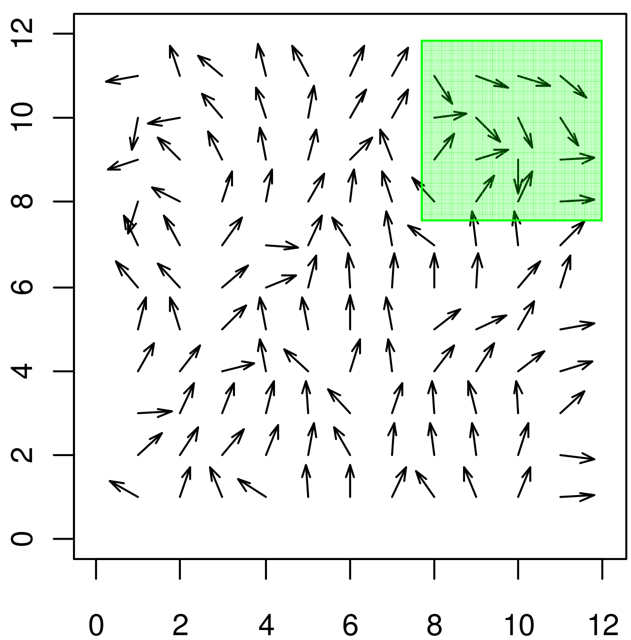

Figure 6-2. Comprehensive Example - Simulated Sample of a Von Mises CRF, $\mu = 0, \rho = \sqrt{0.5}$ with Underlying Trend. Closely related information of the green highlighted area will be enlarged in subsequent figures.

6.3 Estimation of the Spatial Trend

With $\boldsymbol{x}$ the location of a measurement and $\theta(\boldsymbol{x})$ the direction at location $\boldsymbol{x}$, the estimate of the underlying trend was computed by regressing the $\cos(\theta(\boldsymbol{x}))$ and the $\sin(\theta(\boldsymbol{x}))$ on both the horizontal and vertical coordinates of $\boldsymbol{x}$ to avoid the cross over issues of Chapter 2, Section 2.3. The estimates of $\cos(\theta(\boldsymbol{x}))$ and $\sin(\theta(\boldsymbol{x}))$ were combined using the quadrant specific inverse tangent of Chapter 3, Subsection 3.3.1, Equation (3.1). Figure 6-3 compares the trend estimate (tan arrows) vs. the true trend (blue arrows). The blue arrows have the same direction as the blue arrows in Figure 6-1. The trend estimate resembles the true trend. Closely related information of the green highlighted area will be enlarged in subsequent figures.

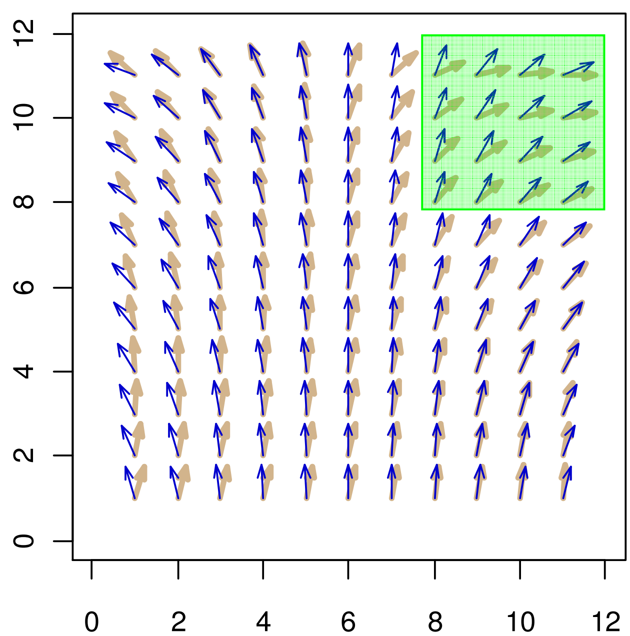

Figure 6-3. Comprehensive Example – Comparison of the Trend Estimate (Tan) with the True Trend (Blue). The trend estimate resembles the true trend. Closely related information of the green highlighted area will be enlarged in subsequent figures.

6.4 Computation of the Residuals

Spatial correlation is encoded in the residuals, which is the angular distance of the observed direction from the spatial trend. The residual at a location is plotted as a unit vector. Its direction equals the observed direction minus the estimated trend direction. The residual is positive [negative] if counterclockwise [clockwise] rotation is required to rotate the trend estimate vector into alignment with the observed direction vector. Figure 6-4 shows the observed direction as black-solid arrows corresponding to Figure 6-2, the trend estimate as tan arrows corresponding to Figure 6-3, and the residual rotation as red-dashed arrows. The plotted area corresponds to the green highlighted area in Figures 6-1 to 6-3. Figure 6-4 was computed using the R package CircSpatial function CircResidual (Appendix J, Section J.3).

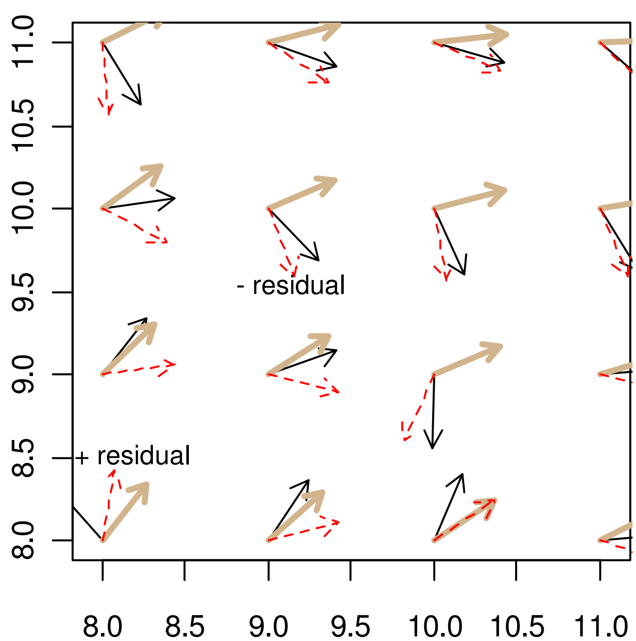

Figure 6-4. Comprehensive Example - Enlarged View of the Observed Direction (Black), Trend Estimate (Tan), and Residual Rotation (Dashed Red) Corresponding to the Green Highlighted Area in Figures 6-1 to 6-3. The residual at a location is plotted as a unit vector with direction equal to the observed direction minus the trend estimate direction. The residual is positive if counterclockwise rotation is required to rotate the trend estimate vector (tan) into alignment with the observed direction vector (black).

6.5 Plotting and Modeling the Cosineogram

Figure 6-5 shows the points of the cosineogram, and the exponential, gaussian, and spherical cosine models of circular-spatial correlation. To determine the best fit of the cosine models, the sill (plateau) was set to 0.674 to approximately center the sill of the models within the cosineogram points on the right. Then, the distance between points of evaluation of the cosineogram was varied to obtain a smooth sequence of points below the range. The range was adjusted for a best overall fit for each model. The spherical cosine model of Chapter 3, Subsection 3.6.2, Equation (3.14), with sill = 0.674 and range $r = 3.07$ was selected for best overall fit. Figure 6-5 was computed using the R package CircSpatial function CosinePlots (Appendix J, Section J.4).

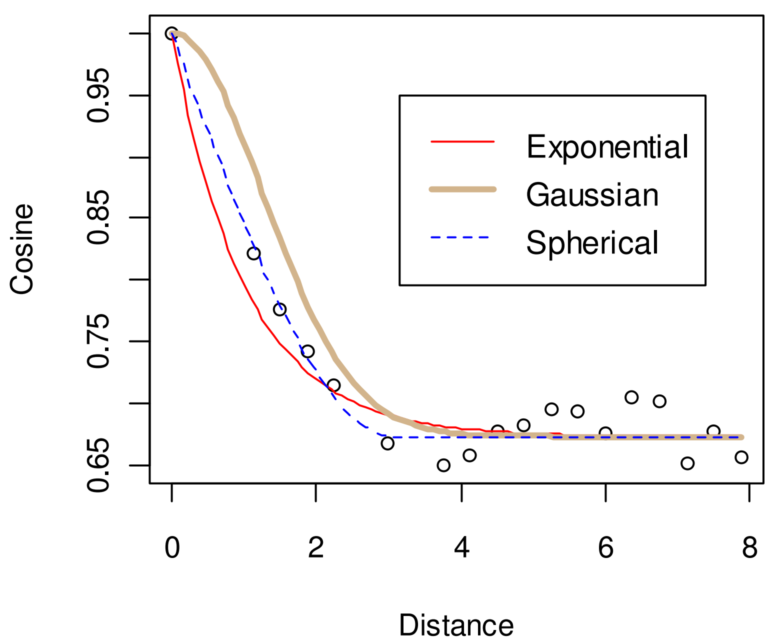

Figure 6-5. Comprehensive Example - Points of the Cosineogram, and the Exponential, Gaussian, and Spherical Cosine Models of Circular-Spatial Correlation. The spherical cosine model with range 3.07 and sill 0.674 was selected for best overall fit to the cosineogram points.

6.6 Kriging the Residuals

The estimates of the random components of direction were computed using the solution of Chapter 4, Subsection 4.2.7, Equation (4.16) with spherical cosine model, range r = 3.07, and sill = 0.674 as determined in Section 6.5. In Figure 6-6 corresponding to the green highlighted area in Figures 6-1 to 6-3, the kriging estimates are plotted as light grey arrows, and the residuals are plotted as red arrows. The red arrows match the direction of the residuals in Figure 6-4. The residuals coincide with the kriging estimates at measurement locations. This was proven in Chapter 4, Section 4.5, and is called “exact interpolation” in linear kriging by the spatial statistics community. Figure 6-6 was computed using the R package CircSpatial function KrigCRF (Appendix J, Section J.5).

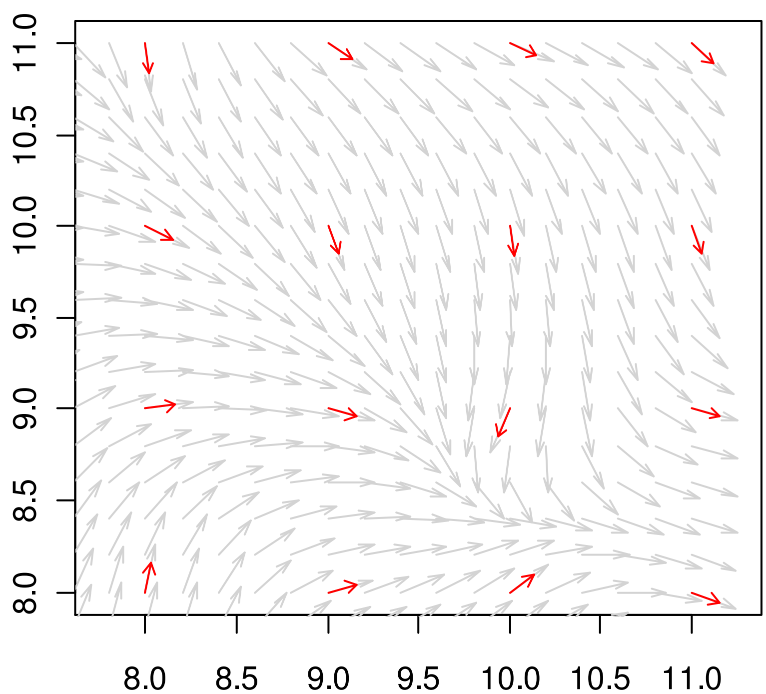

Figure 6-6. Comprehensive Example - Enlarged View of the Kriging (Light Grey) and the Residual Rotations (Red) Corresponding to the Green Highlighted Area in Figures 6-1 to 6-3. The solution of Chapter 4 produces “exact interpolation”, i.e., the kriging equals the residual where the kriging location equals the measurement location.

6.7 Interpolation of the Trend Estimate

To avoid cross over (Chapter 2, Section 2.3), interpolation of the trend estimate is obtained by separately interpolating the cosines and sines of the directions of the trend estimate. A plane is fitted to the three cosine values of the triangular partition of the grid cell of the trend estimate in which an interpolation location occurs. The interpolated cosine is the elevation of the plane at the interpolation location. The sine is interpolated by the same method. The interpolated direction is obtained by applying the quadrant specific inverse tangent of Chapter 3, Subsection 3.3.1, Equation (3.1) to the interpolated sines and cosines. In Figure 6-7, corresponding to the green highlighted area in Figures 6-1 to 6-3, the interpolated direction (purple) matches the direction of the trend estimate (tan) at a measurement location. The tan arrows match the tan arrows in Figures 6-3 and 6-4. Figure 6-7 was computed using the R package CircSpatial function InterpDirection (Appendix J, Section J.6).

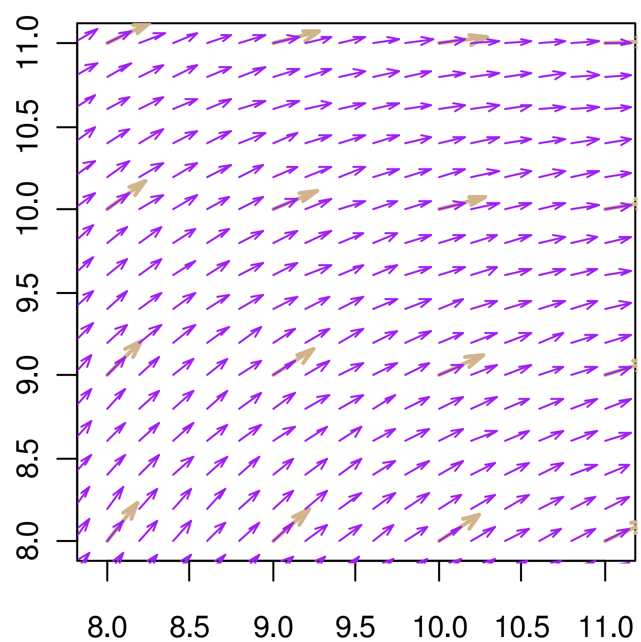

Figure 6-7. Comprehensive Example – Enlarged View of the Interpolation (Purple) of the Trend Estimate (Tan) Corresponding to the Green Highlighted Area in Figures 6-1 to 6-3. The interpolation coincides with the trend estimate at a measurement location.

6.8 Computing The Circular-Spatial Estimate

Figure 6-8, which was constructed using the R code in Appendix L, Section L.9, and corresponds to the green highlighted area in Figures 6-1 to 6-3, shows that the data (black arrows) coincide exactly with the circular-spatial estimate (gold arrows) at sample locations. This is a result of exact interpolation of both the kriging estimate and the spatial trend model. The black arrows match the black arrows in Figures 6-2 and 6-4. The opposing data at (10, 8) and (10, 9) cause the estimates of direction to collide around (10, 8.5).

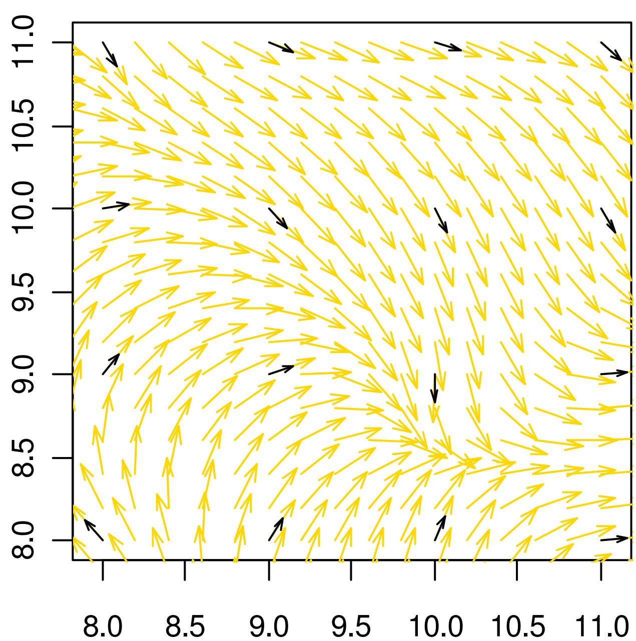

Figure 6-8. Comprehensive Example – Enlarged View of the Circular Spatial Data Estimate (Gold) and the Sample (Black) Corresponding to the Green Highlighted Area in Figures 6-1 to 6-3. At a sample location, the estimate equals the observed direction.

6.9 Imaging the Circular-Spatial Estimate

Figure 6-9 plots the circular-spatial estimate as the circular dataimage of Chapter 2. Figure 6-9 was computed using the R package CircSpatial function CircDataimage (Appendix J, Section J.10) with color wheel rotation = -105º, arrow length multiplier = 0.8, and arrow spacing of one arrow per 3 pixels horizontally and vertically starting at the lower left corner. The black arrows coincide with the black arrows of Figures 6-2, 6-4, and 6-8 where an arrow location coincides with a sample location, e.g., at (8,8).

6.10 Computing the Circular Kriging Variance

The circular kriging variance σ^2_{CK} provides a measure of imprecision of the circular-spatial estimate. $0 \le \sigma^2_{CK} < 4$ (Chapter 4, Section 4.6). $\hat{\sigma}^2_{CK}$, the first order approximation of σ^2_{CK}, was given in Equation (4.26).

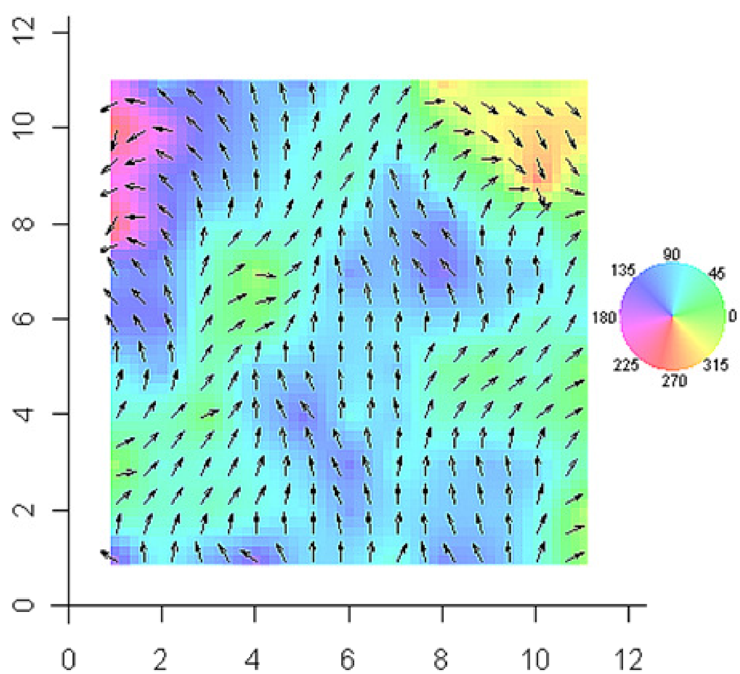

Figure 6-9. Comprehensive Example – Circular Dataimage (Left) of the Circular Spatial Data Estimate with HSV Color Wheel (Right) of Direction.

Figure 6-10 shows $\hat{\sigma}^2_{CK}$ for the estimated circular-spatial data shown in Figure 6-9 corresponding to the measurements on a regular grid shown in Figure 6-2. The measurement locations are indicated by black dots at the center of the green areas. The variability, which is indicated by the linear color scale, decreases as distance to measurement locations decreases. At a measurement location, the estimate equals the data (exact interpolation). Hence, the circular kriging variance is 0 at a measurement location. Figure 6-10 was computed using the R package CircSpatial function KrigCRF (Appendix J, Section J.5). An example of a plot of circular kriging variance with random locations is given in Appendix J, Subsection J.5.4, Figure J-14.

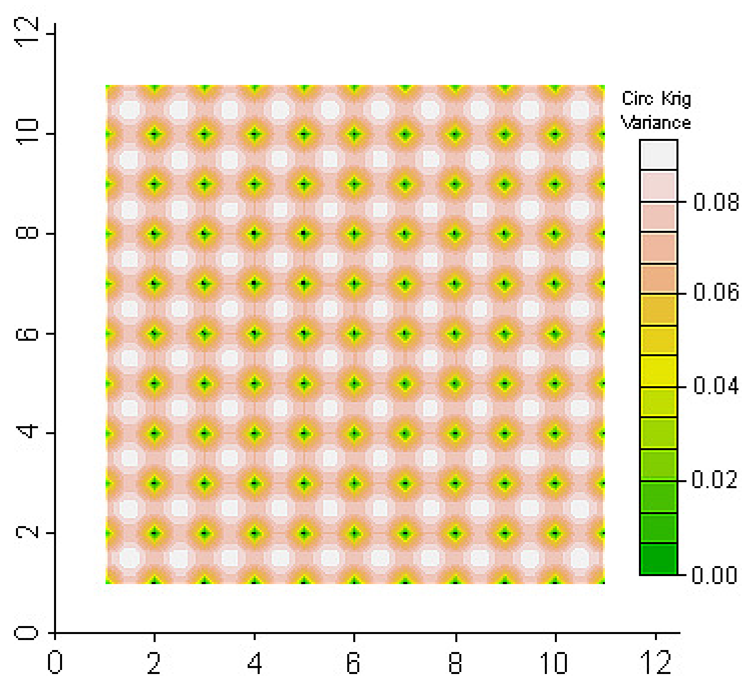

Figure 6-10. Comprehensive Example - Circular Kriging Variance. The circular kriging variance is highly structured with observation locations (black dots) on a regular grid, and decreases to zero as the distance to any measurement location decreases to zero due to “exact interpolation.”